Discipline: **Physics**
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Unit 32:
Lesson/ Module: **Multipole Fields - III**

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Learning Objectives:

From this module, a continuation of the last module, students may get to know about the following:

1. *Relating the multipole radiation to the sources producing the radiation.*
2. *Spherical wave expansion of the Green's function for the Helmholtz equation.*
3. *Equations for the multipole coefficients in terms of sources.*
4. *A look at the dipole case to get some idea of the order of magnitude of the radiation.*
5. *A semiclassical treatment of the radiation from atoms and nuclei.*

32 Multipole Fields - III

32.1 Sources of multipole fields

After having studied the properties of the multipole radiation - the energy, angular momentum and angular distribution etc, we now want to find the radiation produced by specific sources. These sources are the charge density $\rho(\vec{x},t)$ and current density $\vec{J}(\vec{x},t)$. What we have in mind is the application of the ideas to emission of radiation from atoms and nuclei. With this perspective, we also include intrinsic magnetization $\vec{M}(\vec{x},t)$, associated with intrinsic spin. Since we can always analyze the time dependence into its Fourier components, from the very beginning we take the time dependence to be harmonic, so that

$$\rho(\vec{x},t)=\rho(\vec{x})e^{-i\omega t}, \qquad \vec{J}(\vec{x},t)=\vec{J}(\vec{x})e^{-i\omega t}, \qquad \vec{M}(\vec{x},t)=\vec{M}(\vec{x})e^{-i\omega t} \tag{1}$$

We write Maxwell's equations for electric field $\vec{E}$ and $\vec{H}'=\vec{B}/\mu_0$ instead of $\vec{E}$ and $\vec{H}=\vec{B}/\mu$ (outside the sources $\vec{H}'\to\vec{H}$:

$$\vec{\nabla}.\vec{E}=\frac{\rho}{\varepsilon}, \qquad \vec{\nabla}\times\vec{H}'+\frac{ik}{Z_0}\vec{E}=\vec{J}+\vec{\nabla}\times\vec{M} \tag{2}$$

$$\vec{\nabla}.\vec{H}'=0; \qquad \vec{\nabla}\times\vec{E}-ikZ_0\vec{H}'=0; \qquad Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{3}$$

The continuity equation is

$$\vec{\nabla}.\vec{J}=i\omega\rho \tag{4}$$

Let us further change the variable from $\vec{E}$ to $\vec{E}'$ (outside the source $\vec{E}'\to\vec{E}$):

$$\vec{E}'=\vec{E}+\frac{i}{\omega\epsilon_0}\vec{J} \tag{5}$$

This makes the field $\vec{E}'$ also divergenceless. The equations in terms of these quantities become

$$\vec{\nabla}.\vec{E}'=0, \quad \vec{\nabla}\times\vec{H}'+\frac{ik}{Z_0}\vec{E}'=\vec{\nabla}\times\vec{M} \tag{6}$$

$$\vec{\nabla}.\vec{H}'=0, \quad \vec{\nabla}\times\vec{E}'-ikZ_0\vec{H}'=\frac{i}{\omega\epsilon_0}\vec{\nabla}\times\vec{J} \tag{7}$$

The curl equations can be combined to give inhomogeneous Helmholtz equation for $\vec{H}'$ or $\vec{E}'$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}') + \frac{ik}{Z_0} \vec{\nabla} \times \vec{E}' = \vec{\nabla} \times (\vec{\nabla} \times \vec{M})$$

Or

$$\vec{\nabla}(\vec{\nabla}.\vec{H}') - \nabla^2 \vec{H}' + \frac{ik}{Z_0}(ikZ_0 \vec{H}' + \frac{i}{\omega\epsilon_0} \vec{\nabla} \times \vec{J}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{M})$$

Or

$$(\nabla^2 + k^2)\vec{H}' = -\vec{\nabla} \times [\vec{\nabla} \times \vec{M} + \frac{k}{Z_0 \omega \epsilon_0} \vec{J}] = -\vec{\nabla} \times [\vec{\nabla} \times \vec{M} + \vec{J}] \quad (8)$$

Similarly

$$(\nabla^2 + k^2)\vec{E}' = -iZ_0 k \vec{\nabla} \times (\frac{1}{k^2} \vec{\nabla} \times \vec{J} + \vec{M}) \quad (9)$$

These are the counterpart of the homogeneous equations obtained in the earlier units for source-free fields. We solve one of these equations; say the one for $\vec{H}'$ along with $\vec{\nabla}.\vec{H}' = 0$ and use the curl equation $\vec{\nabla} \times \vec{H}' + \frac{ik}{Z_0} \vec{E} = \vec{\nabla} \times \vec{M}$ to find the complete solution of the problem.

The various electric and magnetic multopole coefficients, $a_E(l,m)$ and $a_M(l,m)$

$$a_M(l,m) g_l(kr) = \frac{k}{\sqrt{l(l+1)}} \int Y_l^{m*} (\vec{r}.\vec{H}) d\Omega$$
$$a_E(l,m) f_l(kr) = -\frac{k}{Z_0 \sqrt{l(l+1)}} \int Y_l^{m*} (\vec{r}.\vec{E}) d\Omega \quad (10)$$

involved in the multipole expansion of the fields,

$$\vec{H} = \sum_{l,m} [a_E(l,m) f_l(kr) \vec{X}_l^{\ m} - \frac{i}{k} a_M(l,m) \vec{\nabla} \times (g_l(kr) \vec{X}_l^{\ m})] \quad (11)$$

$$\vec{E} = Z_0 \sum_{l,m} [a_M(l,m) g_l(kr) \vec{X}_l^{\ m} + \frac{i}{k} a_E(l,m) \vec{\nabla} \times (f_l(kr) \vec{X}_l^{\ m})] \quad (12)$$

are given in terms of $\vec{r}.\vec{E}$ and $\vec{r}.\vec{H}$, or equivalently, $\vec{r}.\vec{E}'$ and $\vec{r}.\vec{H}'$. So it is sufficient to write equations for these scalars rather than for the vectors $\overrightarrow{H'}$ and $\overrightarrow{E'}$. For this take the dot product of equations (8) and (9) with $\vec{r}$, use the general vector relations that we encountered earlier also, viz,

$$\nabla^2(\vec{r}.\vec{A}) = 2(\vec{\nabla}.\vec{A}) + \vec{r}.\nabla^2\vec{A}; \qquad \vec{r}.(\vec{\nabla}\times\vec{A}) = (\vec{r}\times\vec{\nabla}).\vec{A} = i\vec{L}.\vec{A} \tag{13}$$

the first on the left hand side and the second on the right hand side in these equations to get the following equations for $\vec{r}.\vec{E}'$ and $\vec{r}.\vec{H}'$:

$$(\nabla^2 + k^2)\vec{r}.\vec{H}' = -i\vec{L}.(\vec{J} + \vec{\nabla}\times\vec{M}) \tag{14}$$

$$(\nabla^2 + k^2)\vec{r}.\vec{E}' = Z_0 k\vec{\mathrm{L}}.(\tfrac{1}{k^2}\vec{\nabla}\times\vec{J} + \vec{M}) \tag{15}$$

We have already solved such inhomogeneous equations via Green's function method (Maxwell's equations-II). The solution depends on the nature of boundary conditions. For outgoing wave boundary condition, the solutions for the Green's function

$$(\nabla^2 + k^2)G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$$

is

$$G(\vec{x},\vec{x}') = -\frac{1}{4\pi}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

So the solutions to the above equations are

$$\vec{r}.\vec{E}'(\vec{x}) = -\frac{Z_0 k}{4\pi}\int d^3x' \frac{e^{ik[\vec{x}-\vec{x}']}}{[\vec{x}-\vec{x}']}\vec{L}'.[\tfrac{1}{k^2}\vec{\nabla}'\times\vec{J}(\vec{x'}) + \vec{M}(\vec{x'})] \tag{16}$$

$$\vec{r}.\vec{H}'(\vec{x}) = \frac{i}{4\pi}\int d^3x' \frac{e^{ik[\vec{x}-\vec{x}']}}{[\vec{x}-\vec{x}']}\vec{L}'.[\vec{\nabla}'\times\vec{M}(\vec{x'}) + \vec{J}(\vec{x'})] \tag{17}$$

Now refer to the unit: Multipole fields-I, where we have defined

$$g_l(kr) = A_l^{(1)}h_l^{(1)}(kr) + A_l^{(2)}h_l^{(2)}(kr) \tag{18}$$

From the asymptotic forms of $h_l^{(1,2)}(kr)$ it follows that since $h_l^{(2)}(kr)$ represents an incoming wave; so the coefficient $A_l^{(2)}$ must be zero. Hence in equations (10) and (11) we take

$$g_l(kr) = f_l(kr) = h_l^{(1)}(kr) \tag{19}$$

outside the sources.

32.2 Spherical wave expansion of the Green's function

We will now find the spherical Green's function for the Helmholtz equation

$$(\nabla^2 + k^2)G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}') \tag{20}$$

For the outgoing wave the Green's function is

$$G(\vec{x},\vec{x}') = \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \tag{21}$$

Similar spherical wave expansion was done for the Poisson equation in the undergraduate classes. Following the same procedure we write the spherical wave expansion for the above Green's function as

$$G(\vec{x},\vec{x}') = \sum_{l,m} g_l(r,r')Y_l^{m*}(\theta',\varphi')Y_l^m(\theta,\varphi) \tag{22}$$

On substituting this expansion for $G(\vec{x},\vec{x}')$ into equation (20), we obtain the following equation for $g_l(r,r')$:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right]g_l(r) = -\frac{1}{r^2}\delta(\vec{x}-\vec{x}') \tag{23}$$

The solution of this equation is given in terms of the spherical Bessel functions. We want the solution that is finite at the origin and an outgoing wave at infinity. Such a solution is given by

$$g_l(r,r') = A_l\, j_l(kr_<)h_l^{(1)}(kr_>) \tag{24}$$

where $(r_<, r_>)$ refer to the (smaller/larger) of r/r'. The correct discontinuity in the slope is assured if $A=ik$, because

$$\frac{dh_l^{(1)}(x)}{dx}j_l(x) - h_l^{(1)}(x)\frac{dj_l(x)}{dx} = \frac{i}{x^2}$$

Hence

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l}Y_l^{m*}(\theta',\varphi')Y_l^m(\theta,\varphi) \tag{25}$$

32.3 Equations for the coefficients

Now look at equations (16) and (17). If we take the point $\vec{x}$ to lie on a spherical surface completely enclosing the source, then $(r_< = r';\ r_> = r)$. On using equation (25) and the orthonormality of the spherical harmonics

$$\int Y_{l'}^{m'*}(\theta,\phi)Y_l^m(\theta,\phi)d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{l'}^{m'*}(\theta,\phi)Y_l^m(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

we have

$$\int d\Omega Y_l^{m*}(\theta,\varphi)\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} = ikj_l(kr')h_l^{(1)}(kr)Y_l^{m*}(\theta',\varphi') \quad (26)$$

By means of this projection we see that $a_M(l,m)$ and $a_E(l,m)$ of equations (12) are given in terms of the integrands of equations (16) and (17) by

$$a_E(l,m) = \frac{ik^3}{\sqrt{l(l+1)}}\int d^3x j_l(kr)Y_l^{m*}(\theta,\varphi)\vec{L}.(\frac{1}{k^2}\vec{\nabla}\times\vec{J}+\vec{M}) \quad (27)$$

$$a_M(l,m) = \frac{-k^2}{\sqrt{l(l+1)}}\int d^3x j_l(kr)Y_l^{m*}(\theta,\varphi)\vec{L}.(\vec{\nabla}\times\vec{M}+\vec{J}) \quad (28)$$

Given the source densities $\vec{J}$ and $\vec{M}$, these equations provide us the strength of the multipole radiations, $a_M(l,m)$ and $a_E(l,m)$. The equations can be put in an alternative and more transparent form by using the definition of the angular momentum operator and various vector relations. Thus for any vector field $\vec{A}$

$$\vec{L}.\vec{A} = i\vec{\nabla}.(\vec{r}\times\vec{A}) \quad (29)$$

$$\vec{L}.(\vec{\nabla}\times\vec{A}) = i\nabla^2(r.\vec{A}) - \frac{i}{r}\frac{\partial}{\partial r}(r^2\vec{\nabla}.\vec{A}) \quad (30)$$

On using the first identity for $\vec{M}$ and the second for $\vec{J}$, we obtain for $a_E(l,m)$

$$a_E(l,m) = -\frac{k^3}{\sqrt{l(l+1)}}\int d^3x j_l(kr)Y_l^{m*}(\theta,\varphi)[\frac{1}{k^2}\nabla^2(\vec{r}.\vec{J}) - \frac{ic}{k}\frac{1}{r}\frac{\partial}{\partial r}(r^2\rho) + \vec{\nabla}.(\vec{r}\times\vec{M})] \quad (31)$$

In the last step we have used the continuity equation (4) to express $\vec{J}$ in terms of charge density ρ. We now use the Green's theorem

$$\int_V (\varphi\nabla^2\psi - \psi\nabla^2\varphi)dV = \int_S (\varphi\vec{\nabla}\psi - \psi\vec{\nabla}\varphi).\hat{n}da$$

For infinite volume, the surface integral on the right vanishes and we have

$$\int_V \varphi\nabla^2\psi dV = \int_V \psi\nabla^2\varphi dV$$

If we apply this on the first term in the square brackets of equation (31) with $\varphi = j_l(kr)$, $\psi = (\vec{r}.\vec{J})$, then ∇^2 is replaced by $-k^2$. In the second term we do radial integration by parts (again neglecting the surface term):

$$\int r^2 dr j_l(kr) \frac{1}{r}\frac{\partial}{\partial r}(r^2\rho) = -\int dr r^2 \rho \frac{\partial}{\partial r}\{rjk_l(kr)\}$$

The result is

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int d^3x Y_l^{m*}(\theta,\varphi)[ik(\vec{r}.\vec{J})\,j_l(kr) + c\rho\frac{\partial}{\partial r}\{rj_l(kr)\} - ik\vec{\nabla}.(\vec{r}\times\vec{M})\,j_l(kr)] \tag{32}$$

In a similar fashion

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int d^3x Y_l^{m*}(\theta,\varphi)[\vec{\nabla}.(\vec{r}\times\vec{J})\,j_l(kr) + (\vec{\nabla}.\vec{M})\frac{\partial}{\partial r}\{rj_l(kr)\} - k^2(\vec{r}.\vec{M})\,j_l(kr)] \tag{33}$$

32.3.1 Further Simplification

These results are valid for any source size and any frequency. However, for many applications, particularly in nuclear and atomic systems, the source dimensions are very small compared to the wavelength. If r represents the size of the source, then $kr \ll 1$. In such a situation multipole coefficients can be simplified considerably. We can replace $j_l(kr)$ by its small argument limit:

$$j_l(x) \to \frac{x^l}{1.3.5\ldots(2l+1)}\left(1 - \frac{x^2}{2(2l+3)} + \ldots\right)$$

If we keep only the lowest order term in (kr), the above expression for the electric multipole coefficients is approximated to

$$a_E(l,m) = \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}(Q_{lm} + Q'_{lm}) \tag{34}$$

The multipole moments Q_{lm}, Q'_{lm} are

$$Q_{lm} = \int r^l Y_l^{m*}\rho d^3x \tag{35}$$

$$Q'_{lm} = \frac{-ik}{(l+1)c}\int r^l Y_l^{m*}\vec{\nabla}.(\vec{r}\times\vec{M})d^3x \tag{36}$$

The Q_{lm} is the same as the electric multipole moment defined in electrostatics except of course that now the charge density is a function of time. The Q'_{lm} is the *induced* electric multipole moment due to intrinsic magnetization. It is generally much smaller than Q_{lm}. The electric multipole radiation is thus essentially determined by the charge density of a distribution.

If we apply the same "long wavelength" limit $kr<<1$ to the coefficients $a_M(l,m)$, and follow the same procedure, we obtain the following approximate expression for these coefficients:

$$a_M(l,m) \cong \frac{ik^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}(M_{lm}+M'_{lm}) \tag{37}$$

The M_{lm} and M'_{lm} are the magnetic multipole moments and are given by

$$M_{lm} = \frac{-1}{(l+1)}\int r^l Y_l^{m*}\vec{\nabla}.(\vec{r}\times\vec{J})d^3x \tag{38}$$

$$M'_{lm} = -\int r^l Y_l^{m*}\vec{\nabla}.\vec{M}d^3x \tag{39}$$

If a system has intrinsic magnetic moment, the two terms above are of the same order of magnitude. The magnetic multipole radiation is determined by the magnetic moment densities, $(\vec{r}\times\vec{J})/2$ and $\vec{M}$.

32.3.2 The dipole case

Let us look at the special case of dipole emission. Let us first look at the contribution due to the charge density $\rho(\vec{r})$ and current density $\vec{J}(\vec{r})$ only. Further consider first the special case of radiation in the *m*=0 case. In fact this can be considered the general case, since the other two, $m = \pm 1$ are related to *m*=0, by rotation. From equation (35) we get

$$Q_{10} = \int rY_1^0\rho d^3x = \sqrt{\frac{3}{4\pi}}\int r\rho\cos(\theta)d^3x = \sqrt{\frac{3}{4\pi}}\int z\rho d^3x\,;$$

Similarly from equation (38)

$$M_{10} = \frac{-1}{2}\int rY_1^0\vec{\nabla}.(\vec{r}\times\vec{J})d^3x = -\frac{1}{2}\sqrt{\frac{3}{4\pi}}\int z\vec{\nabla}.(\vec{r}\times\vec{J})d^3x$$

This equation can be further simplified by partial integration. The result is

$$M_{10} = \frac{1}{2}\sqrt{\frac{3}{4\pi}}\int (xJ_y - yJ_x) d^3x$$

The current density $\vec{J}$ is of order $\vec{v}\rho$, where $\vec{v}$ is the velocity of motion of the charges. Hence from the definitions (34) and (37) of the amplitudes a_E and a_M respectively, a_M is smaller than a_E by a factor of order (v/c) and hence the intensity by a factor of order $(v/c)^2$. Though we have proved this result only for dipole radiation, it is in fact true for any multipole: For a multipole of any order l, the intensity of the magnetic radiation is order $(v/c)^2$ smaller than the electric radiation.

The contribution of magnetization to the radiation is obtained from equations (36) and (39) respectively. For the dipole case, simplification by partial integration as before leads to

$$Q'_{10} = \frac{ik}{2c}\sqrt{\frac{3}{4\pi}}\int (\vec{r}\times\vec{M})_z d^3x$$

$$M'_{10} = \sqrt{\frac{3}{4\pi}}\int M_z d^3x$$

32.4 Multipole radiation in atoms and nuclei

Radiation is produced in transitions from one quantum state of an atom or a nucleus to another. So basically multipole radiation in atoms and nuclei is a quantum process and requires the use of quantum mechanics. There are two distinct steps involved in this quantization process. One is that light is not emitted in a continuous process but as distinct quanta of energy $\hbar\omega$. Secondly, the source of the radiation is a quantum mechanical system rather than a classical distribution of currents and charges. However some qualitative aspects can be understood from a semiclassical treatment on the application of formulas derived above.

Transition probability

The transition probability, Γ (which is the reciprocal of the mean life of transition) for emission of a photon of energy $\hbar\omega$, is given by the radiated power divided by $\hbar\omega$. The expression for the radiated energy was derived in the last module, and is

$$P(l,m) = \frac{\mu_0 c}{2k^2}|a(l,m)|^2 \tag{40}$$

In the long wavelength limit, the amplitudes a_E and a_M are given by equations (34) and (37) respectively. These lead to the following expression for the transition probability for the electric dipole case

$$\Gamma_E(l,m) = \frac{\omega Z_0 k^{2l}}{2\hbar[(2l+1)!!]^2}\frac{l+1}{l}|Q_{lm} + Q'_{lm}|^2 \tag{41}$$

For magnetic multipole transitions the expression is the same except for the replacement

$$Q_{lm} + Q'_{lm} \rightarrow (\frac{1}{c})(M_{lm} + M'_{lm}) \tag{42}$$

To find the order of the effective multipole moments we proceed as follows: In the definition (35) of Q_{lm}, the spherical harmonic is of order unity. The integral of the charge density $\rho(r)$ is order e, the effective charge. If the effective size of the system is R, then Q_{lm} is of order

$$|Q_{lm}| = O(eR^i) \tag{43}$$

Further, if the effective mass of the system is m then the effective magnetic moment of the constituents is $\frac{e\hbar}{m}$ and the effective magnetization is

$$\vec{M} = O(\frac{e\hbar}{mR^3}) \tag{44}$$

The most naïve estimates of multipole coefficient Q'_{lm} is then

$$|Q'_{lm}| = O(\frac{\hbar\omega}{mc^2}eR^l) \tag{45}$$

Similarly

$$\frac{1}{c}(M_{lm} + M'_{lm}) = O(\frac{\hbar}{mc}eR^{l-1}) \tag{46}$$

Using these rough estimates, we can study some qualitative features of radiative transitions in atoms and nuclei. Both in atoms and nuclei, the transition energies, $\hbar\omega$ are usually small compared to the rest energy mc^2 of the system. This implies that we can expect

$$Q'_{lm} << Q_{lm}. \tag{47}$$

Thus the electric multipole transitions are dominated by the charge density with small contribution from intrinsic magnetization. On the other hand, the magnetic multipole transitions get similar contributions from orbital magnetization and intrinsic magnetization due to spin.

In atoms it is the electrons that are the source of radiation. The size of the system depends on whether radiation is being emitted by the valence electrons or by the inner orbit electrons. If a_0 is the radius of the Bohr atom, the size of the radiating system can be written as $R = O(\frac{a_0}{Z_{eff}})$. Then Z_{eff} is of order unity for valence electron transitions, and of order Z for inner shell electron x-ray transitions.

From equations (43) and (46) above, the relative amplitude of the magnetic multipole moment to the electric multipole moment of the same order is

$$\frac{|M|}{c|Q|} = O(\frac{\hbar}{mcR}) = O(\frac{\hbar Z_{eff}}{mca_0}) = O(\frac{Z_{eff}}{137})$$

Hence the transition probabilities will be in the ratio

$$\frac{\Gamma_M(l)}{\Gamma_E(l)} = O[(\frac{Z_{eff}}{137})]^2$$

For valence electron transitions this ratio is

$$O(\frac{1}{137})^2$$

Hence magnetic transitions are completely negligible compared to the electric transitions of the same order. For x-ray transitions in heavy elements $Z_{eff} \approx Z$. Hence $(\frac{Z_{eff}}{137})^2$ is still small but not negligible. Only in this situation are the magnetic transitions of any significance. However there is an overriding clause due to the selection rules. If the lowest order electric transition is forbidden but the corresponding magnetic transition is allowed by the selection rules, then such a transition may also become important.

The relative size of transitions, electric or magnetic, differing in order by one unit is also of interest. From equations (41)-(46)

$$\frac{\Gamma(l+1)}{\Gamma(l)} = O(kR)^2$$

In atoms, the transition energies are $O(Z_{eff}{}^2 mc^2/137^2)$. The radius is

$$R = O(a_0 / Z_{eff}) = O(\frac{137\hbar}{mcZ_{eff}}) \Rightarrow kr = O(Z_{eff} / 137) \Rightarrow \frac{\Gamma(l+1)}{\Gamma(l)} = O(Z_{eff} / 137)^2 = O(\frac{\Gamma_M(l)}{\Gamma_E(l)})$$

Thus if the selection rules allow several multipoles, the lowest allowed order generally dominates. Also the ratio of the magnetic to electric transition probability is of the same order as the ratio of probability of (l+1) to l transition. Thus if the lowest allowed magnetic transition is of order l but electric is of order (l+1), the two can be of same order. But in the opposite case the magnetic transition is completely negligible.

In nuclei also the estimate $\frac{\Gamma(l+1)}{\Gamma(l)} = O(Z_{eff} / 137)^2 = O(\frac{\Gamma_M(l)}{\Gamma_E(l)})$ holds true. Numerically, the nuclear radius $R \approx 1.4 \times 10^{-15} A^{1/3} \Rightarrow (kR) \approx 1.4 \times 10^{-15} \frac{\hbar\omega}{\hbar c} A^{1/3}$. This yields $kR \approx \frac{\hbar\omega(MeV)}{140} A^{1/3}$. Transition energies in nuclei vary over a wide range – from several keV to several MeV. This means that for heavy nuclei $kR \approx 0.1 - 0.0001.$ Thus for high energy transitions the suppression factor for higher order transitions is not very large. However, for low energy transitions higher order multipoles are highly suppressed.

Summary

1. *After having studied the properties of multipole radiations we next wish to relate them to the sources of these radiations.*
2. *Since, we have in mind radiation from atoms and nuclei, the intrinsic magnetization of atoms and nuclei due to spin angular momentum is explicitly included in this study.*
3. *To relate the sources to the coefficients in the multipole expansion, we obtain spherical wave solutions of the Helmholtz equation.*
4. *We next look at the special case of multipole radiation and conclude that the magnetic dipole radiation is weaker than the electric dipole radiation by a factor of* $(v/c)^2$*. This is in fact true for multipoles of all orders.*
5. *Lastly we briefly look at the multipole radiation in atoms and nuclei. This subject requires quantum mechanics for its proper treatment. So all we do is to make some general comments on radiation in atoms and nuclei.*